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q-Sigma-Algebra Generated by Balls

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The q-σ-algebra, i.e., the system of sets closed under complementation, countable disjoint unions, and containing the empty set, generated by the system of open balls coincides with the σ-field of Borel sets in $\mathbb{R}^n$ for $n = 1$, 2, and 3. A first step to extend the proof for $n = 4$, 5, 6, and 7 is indicated.

Let X be an arbitrary nonempty set. A class $\mathscr{L}$ of subsets of the set X, containing the empty set, is said to be *a q-σ-algebra* [*a concrete quantum logic* (Pták, Pulmannová, 1991, p. 2)] if it is closed with respect to complementation and with respect to the union of any sequence of pairwise disjoint sets. If a *σ-algebra* of sets is defined in the usual way as a class of sets containing the empty set, closed with respect to complementation and with respect to unions of arbitrary sequences of sets, then, obviously, for an arbitrary class $\mathscr{C}$ of subsets of X, the q-σ-algebra $\mathscr{L}(\mathscr{C})$ generated by $\mathscr{C}$ is contained in the σ-algebra $\mathscr{A}(\mathscr{C})$ generated by $\mathscr{C}$. It is known (Neubrunn, 1970, Corollary 1) that if $\mathscr{C}$ is closed with respect to intersection, i.e., if $A \in \mathscr{C}$, $B \in \mathscr{C}$ implies $A \cap B \in \mathscr{C}$, then $\mathscr{L}(\mathscr{C}) = \mathscr{A}(\mathscr{C})$. Therefore, for instance, if $\mathscr{C}$ is the class of all (open) intervals on the real line, then $\mathscr{L}(\mathscr{C}) = \mathscr{A}(\mathscr{C})$, where $\mathscr{A}(\mathscr{C}) = \mathscr{B}$ is the class of Borel sets. For the same reason an analogous equality holds in the plane. In fact, if $\mathscr{C}$ is the class of all (open) rectangles, then $\mathscr{L}(\mathscr{C})$ coincides with the class $\mathscr{B}$ of all Borel sets in the plane. However, if $\mathscr{C}$ is taken to be the set of all discs (the fact they are open, closed, or both is not essential), the question arises of whether $\mathscr{L}(\mathscr{C})$ contains all Borel sets, i.e., whether $\mathscr{L}(\mathscr{C}) = \mathscr{B}$. More generally: *Let $\mathscr{C}$ be the set of* (*n-dimensional*) *open balls in the Euclidean space* $\mathbb{R}^n$. *Does* $\mathscr{L}(\mathscr{C})$ *equal* $\mathscr{B}$? The question has been raised in much more general form (for Banach algebras) by Preiss.

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However, it appeared to be nontrivial even in the above description, which was formulated independently by Neubrunn (1977). The problem was positively solved in Olejček (1988) for the 2-dimensional space and in Olejček (1995) for the 3-dimensional space.

The method of the proof used in the 3-dimensional space (which can be applied also in the 2-dimensional space) is based on a cover of the unit cube by four disjoint sets, which are constructed using set operations on balls permitted within the q-σ-algebra. The first step of the construction is to cover the unit cube by the system of four mutually orthogonal balls. The *orthogonality* is meant geometrically, i.e., two balls are called *orthogonal* if the square of the distance of their centers equals the sum of the squares of their radii. If a ball in an n-dimensional space is described by an $(n + 1)$-dimensional vector $[x_1, x_2, \ldots, x_n; r]$, where x_i is the ith coordinate of the center and r is the radius, then the unit cube in the 3-dimensional space with vertices [0, 0, 0], [0, 0, 1], [0, 1, 0], [0, 1, 1], [1, 0, 0], [1, 0, 1], [1, 1, 0], and [1, 1, 1] is covered by the balls [0, 0, 1; 1], [0, 1, 0; 1], [1, 0, 0; 1], and [1, 1, 1; 1].

In this paper we try to find a method for construction of a similar cover in higher dimensions. In fact, for our purpose, the radii of the balls should not exceed one and not all balls in the cover have to intersect each other. Summarized, we try to find a finite cover of the unit cube in the n-dimensional space by closed balls with radii not exceeding one, which are in one of the following mutual positions: disjoint, tangential, or orthogonal.

Due to symmetry of the unit cube we try to find a symmetrical cover. It can be expressed in a form reduced with respect to permutations. For example, in the 3-dimensional space the reduced system representing the cover is [0, 0, 1; 1], [1, 1, 1; 1]. All other balls of the cover can be obtained by permutations.

In the 4-dimensional space the pattern from the 3-dimensional space can be applied, i.e., the cover is constructed by unit balls situated in the vertices with an odd sum of the coordinates. It consists of [0, 0, 0, 1; 1], [0, 0, 1, 0; 1], [0, 1, 0, 0; 1], [1, 0, 0, 0; 1], [0, 1, 1, 1; 1], [1, 0, 1, 1; 1], [1, 1, 0, 1; 1], and [1, 1, 1, 0; 1]. In the reduced form the cover is represented by [0, 0, 0, 1; 1] and [0, 1, 1, 1; 1].

In the 5-dimensional space the situation is slightly more complicated. The system of unit balls with centers in the vertices with an odd sum of coordinates does not cover the unit cube. However, if the ball $[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; \frac{1}{2}]$ is added, the cover is complete. In the reduced form it consists of [0, 0, 0, 0, 1; 1], [0, 0, 1, 1, 1; 1], [1, 1, 1, 1, 1; 1], and $[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; \frac{1}{2}]$.

The idea of construction of a cover can be extended to higher dimensions by means of generalized spherical (or circle) inversion *inv* determined by a sphere (or circle) situated in the origin with radius $\sqrt{2}$. It is a transformation defined by

$$y_i = \frac{2x_i}{\sum_{j=1}^{n} x_j^2}$$

for $i = 1, 2, \ldots, n$. It is easy to check that such a transformation transforms a ball $[x_1, x_2, \ldots, x_n; r]$ onto a ball $[y_1, y_2, \ldots, y_n; s]$, where

$$y_i = \frac{2x_i}{\sum_{j=1}^{n} x_j^2 - r^2}, \qquad s = \frac{2r}{\sum_{j=1}^{n} x_j^2 - r^2}$$

Note three properties of the transformation *inv*: it is symmetric, orthogonality preserving, and an involution. In the 5-dimensional space the following pairs are mutual images:

$$[0, 0, 0, 0, 1; 1] \leftrightarrow \text{hyperplane } x_5 = 1$$

$$[0, 0, 1, 1, 1; 1] \leftrightarrow [0, 0, 1, 1, 1; 1]$$

$$[1, 1, 1, 1, 1; 1] \leftrightarrow [\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}; \tfrac{1}{2}]$$

This explains why the overlapping balls are mutually orthogonal.

Let us apply the similar construction in the 6-dimensional space. First we take the unit balls in the vertices with an odd sum of coordinates. In the reduced version we obtain [0, 0, 0, 0, 0, 1; 1], [0, 0, 0, 1, 1, 1; 1], [0, 1, 1, 1, 1, 1; 1]. Then we transform the third one to obtain $[0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; \frac{1}{2}]$. However, the cover is still not complete. The reason is that the system is not symmetric within the reduced form. Namely the ball $[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1; \frac{1}{2}]$ is missing. It transforms to itself in *inv* and, if included, the cover is complete.

The algorithm applied can be generalized in the following way:

(1) List the system of unit balls situated in the corners of the unit cube with an odd sum of coordinates.

(2) Add all of their images in *inv* interfering with the unit cube.

(3) Complete the system with respect to the symmetry.

(4) Repeat steps 2 and 3 until nothing new is obtained.

In 7-dimensional space this produces the following:

(1) [0, 0, 0, 0, 0, 0, 1; 1], [0, 0, 0, 0, 1, 1, 1; 1], [0, 0, 1, 1, 1, 1, 1; 1], [1, 1, 1, 1, 1, 1, 1; 1].

(2) $[0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; \frac{1}{2}]$, $[\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}; \frac{1}{3}]$.

(3) $[0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1; \frac{1}{2}]$, $[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1, 1; \frac{1}{2}]$, $[\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}; \frac{1}{3}]$, $[\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}; \frac{1}{3}]$, $[\frac{1}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}; \frac{1}{3}]$.

(4-2) $[\frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; \frac{1}{4}]$.
(4-3) $[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{4}; \frac{1}{4}]$.

And the system is closed.

Unfortunately, the algorithm does not produce any finite cover for dimensions $n > 7$ and the problem remains open.

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